6. Yu. V. Baryshev, A. I. Leont'ev, N. K. Peiker, and V. I. Rozhdestvenskii, "The effect of a longitudinal pressure gradient on the effectiveness of a gas curtain in a subsonic turbulent boundary layer," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 2 (1975).
7. J. J. Nicolas and M. Izard, "Protection thermique de tuyéres supersoniques par film gazeux," La Recherche Aérospatiale, No. 4 (1971).
8. S. S. Kutateladze and A. I. Leont'ev, Heat and Mass Transfer and Friction in a Turbulent Boundary Layer [in Russian], Izd. Énergiya, Moscow (1972).
9. R. A. Seban, "Heat transfer and effectiveness for a turbulent boundary layer with tangential fluid injection," Trans. ASME, Ser. C, 82, No. 4 (1970).
10. G. N. Abramovich, The Theory of Turbulent Jets [in Russian], Izd. Fizmatgiz, Moscow (1960).

STABILITY OF A PLANE JET IN A MEDIUM WITH RELAXATION

V. V. Sobolev and O. Yu. Tsvelodub

UDC 532.5

1. Presentation of the Problem and Basic Equations. Let us consider the stability (relative to infinitely small perturbations) of the steady-state jet flow of a liquid having the following equation of state [1, 2]:

$$\delta p = c_0^2 \delta\rho + \beta \frac{d}{dt} \delta\rho + \varkappa \frac{d^2}{dt^2} \delta\rho, \tag{1.1}$$

where δp and $\delta\rho$ are small perturbations of pressure and density; c_0 is the velocity of sound in the medium; and β and $\varkappa$ are the relaxational viscosity and dispersion coefficients. A detailed derivation of the equations was given in [2, 3] for perturbations of the velocity v and pressure p. If we express the perturbed quantities in the form

$$f(x, y, z, t) = f(y) \exp\,[i\alpha(x-ct) + i\gamma z],$$

where f is the perturbation of the pressure, density, or velocity components; x, y, z are the spatial coordinates; α, γ are the wave numbers; and c is the velocity ($c = c_r + ic_i$); the equations for the two-dimensional perturbations v(y) and p(y) take the form [3]

$$v'' - V'(V-c)Av' - \left(B + \frac{V''}{V-c} - AV'^2\right)v = 0; \tag{1.2}$$

$$p'' - \frac{2V'}{V-c}p' - Bp = 0,$$

$$A = \frac{M^2(2 + i\alpha\beta M^2(V-c))}{(1 + i\alpha\beta M^2(V-c) - \varkappa\alpha^2 M^2(V-c)^2)(M^2(1+\varkappa\alpha^2)(V-c)^2 - i\alpha\beta M^2(V-c) - 1)}, \tag{1.3}$$

$$B = \alpha^2(1 - M^2(V-c)^2/(1 + i\alpha\beta M^2(V-c) - \varkappa\alpha^2 M^2(V-c)^2)),$$

where V is the velocity profile of the main flow; M is the Mach number ($M = V_{max}/c_0$). In this paper we have

$$V = 1/\mathrm{ch}^n y, \tag{1.4}$$

where n is a natural number.

The problem as to the stability of steady-state flow (1.4) reduces to a determination of the eigenvalues of c for the equations (1.2), (1.3). The stability may be studied on the basis of Eqs. (1.2), (1.3) for two-dimensional perturbations, since it may be shown that the problem of stability relative to three-dimensional perturbations is equivalent to the problem of stability relative to two-dimensional perturbations with a smaller Mach number and a larger parameter β.

The boundary conditions for Eqs. (1.2), (1.3) involve the requirement that v and p should be finite at $y = \pm\infty$.

Let us consider some relationships for c. The semicircular theorem limiting the range of unstable eigenvalues for parallel flows in an incompressible stratified liquid was proved in [4]. Let us consider the conditions

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 73-78, March-April, 1977. Original article submitted March 29, 1976.

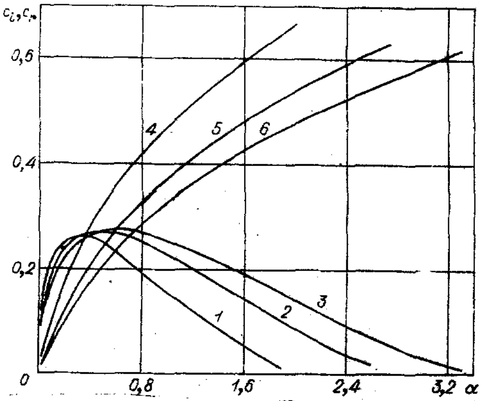

Fig. 1

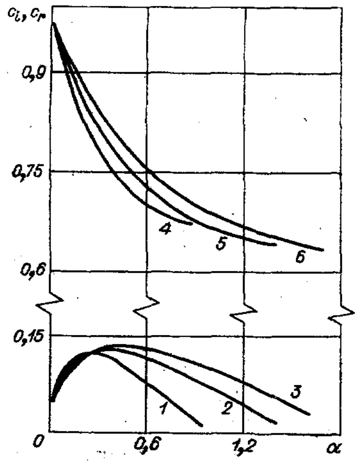

Fig. 2

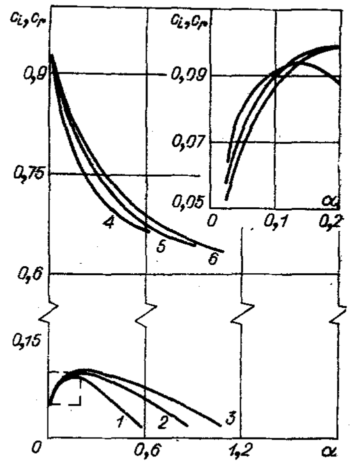

Fig. 3

TABLE 1

α	Mode I, M=0	Mode I, M=1	Mode II, M=0	Mode II, M=1
	c_i			
0,025	0,09371	0,09390	0,03963	0,05363
0,05	0,12944	0,12987	0,05920	0,07016
0,1	0,17478	0,17555	0,08488	0,08765
0,125	0,19101	0,19183	0,09406	0,09270
0,2	0,22589	0,22628	0,11313	0,10041
0,3	0,25336	0,25213	0,12642	0,10109

TABLE 2

α	1	2	3
0,1	0,05356	0,05356	0,05354
	0,17588	0,17588	0,17587
0,5	0,22293	0,22297	0,22188
	0,26634	0,26637	0,26730
0,6	0,25388	0,25395	0,25262
	0,26434	0,26438	0,26588
0,8	0,30615	0,30638	0,30473
	0,25091	0,25092	0,25376
1,0	0,34895	0,34943	0,34761
	0,23071	0,23059	0,23487
1,5	0,43166	0,43319	0,43084
	0,16999	0,16897	0,17690
2,0	0,49772	0,50156	0,49682
	0,10828	0,10546	0,11712
2,4	0,54645	0,55522	0,54454
	0,06319	0,05907	0,07286
2,6	0,57033	0,58332	0,56763
	0,04292	0,04122	0,05264

TABLE 3

α	1	2	3
0,1	0,82296	0,82270	0,83607
	0,05227	0,05184	0,04578
0,2	0,77524	0,77464	0,79174
	0,06959	0,06857	0,06183
0,3	0,73948	0,73851	0,75938
	0,07049	0,06873	0,06303
0,4	0,71152	0,71018	0,73411
	0,06497	0,06244	0,05865
0,6	0,67109	0,66888	0,69715
	0,04549	0,04129	0,04245
0,8	0,64493	0,64407	0,67373
	0,02226	0,01848	0,02067

under which this theorem holds for a liquid with an equation of state (1.1) as well. Let $c_i \neq 0$. Dividing (1.3) by $(V - c)^2$, we obtain

$$[(V - c)^{-2}p']' - \alpha^2[(V - c)^{-2} - M^2/D^{-1}]p = 0,$$
$$D = 1 + i\alpha\beta M^2(V - c) - \varkappa\alpha^2 M^2(V - c)^2. \tag{1.5}$$

Multiplying (1.5) by p^*, we integrate with respect to y using the boundary conditions and separate the real and imaginary parts in the resultant expression:

$$\int [(V - c_r)^2 - c_i^2]\left(\frac{|p'|^2 + \alpha^2|p|^2}{|V - c|^4} + \varkappa\alpha^4 M^4 |p/D|^2\right) dy - \alpha^2 M^2 \int (1 + \alpha\beta M^2 c_i)\,|p/D|^2\, dy = 0; \tag{1.6}$$

$$\int (V - c_r)\left[2c_i\left(\frac{|p'|^2 + \alpha^2|p|^2}{|V - c|^4} + \varkappa\alpha^4 M^4 |p/D|^2\right) + \alpha^3\beta M^4 |p/D|^2\right] dy = 0. \tag{1.7}$$

It follows from (1.7) that for growing perturbations with $\varkappa \geq 0$ and $\beta \geq 0$ a point y_c exists such that $c_r = V(y_c)$. We see that for $\varkappa \geq 0$, $\beta = 0$ Eqs. (1.6) and (1.7) have the same form as the analogous expressions in [4]. Thus, in the present case we have

$$\left\{\left[c_r - \frac{1}{2}(V_{\max} + V_{\min})\right]^2 + c_i^2 - \frac{1}{4}(V_{\max} - V_{\min})^2\right\}\int Q dy + \alpha^2 M^2 \int \frac{|p|^2 dy}{|1 - \varkappa\alpha^2 M^2 (V - c)^2|^2} \leqslant 0; \quad (1.8)$$

$$Q = \frac{|p'|^2 + \alpha^2 |p|^2}{|V - c|^4} + \frac{\varkappa\alpha^4 M^4 |p|^2}{|1 - \varkappa\alpha^2 M^2 (V - c)^2|^2} \geqslant 0. \quad (1.9)$$

It follows from (1.8) and (1.9) that

$$\left[c_r - \frac{1}{2}(V_{\max} + V_{\min})\right]^2 + c_i^2 \leqslant \frac{1}{4}(V_{\max} - V_{\min})^2. \quad (1.10)$$

Equation (1.10) defines the eigenvalues c of the problem as to the Rayleigh stability of parallel flows in a medium with an equation of state (1.1). In the particular case of jet flow (1.4), the inequality (1.10) takes the form

$$\left(c_r - \frac{1}{2}\right)^2 + c_i^2 \leqslant \frac{1}{4}.$$

2. Study of Jet Stability. Since the profile of the main flow (1.4) is an even function, we may seek the solutions for perturbations symmetrical with respect to the velocity v (antisymmetrical with respect to the pressure p) and antisymmetrical with respect to v (symmetrical with respect to p) separately.

After making the substitutions

$$z = \text{th}\, y; \quad \varphi = p'/p; \quad \psi = v'/v \quad (2.1)$$

the homogeneous linear equations of the second order (1.2), (1.3) reduce to the nonlinear Riccati equations:

$$(1 - z^2)\varphi' + \varphi^2 - \frac{V'}{V - c}\varphi - B = 0; \quad (2.2)$$

$$(1 - z^2)\psi' + \psi^2 - V'(V - c)A\psi - B - \frac{V''}{V - c} + AV'^2 = 0,$$
$$V = (1 - z^2)^{n/2}, \quad V' = -nzV,$$
$$V'' = nV((n + 1)z^2 - 1). \quad (2.3)$$

The primes in (2.2), (2.3) signify derivatives with respect to z. The boundary conditions for φ and ψ at the point $z = -1$ have the form

$$\varphi(-1) = \psi(-1) = \alpha\left(1 - \frac{M^2 c^2}{1 - i\alpha\beta M^2 c - \varkappa\alpha^2 M^2 c^2}\right)^{1/2}. \quad (2.4)$$

We see from (2.2)-(2.4) that the values of the derivatives φ' and ψ' at this point are not properly determinate. Resolving the indeterminacy by the L'hôpital rule, we obtain the boundary values of φ' and ψ'. For $n > 2$

$$\varphi'(-1) = \psi'(-1) = 0;$$

for $n = 2$

$$\varphi'(-1) = (\Delta - 4\varphi(-1)/c)/(1 + \varphi(-1));$$

$$\psi'(-1) = \left(\Delta - \frac{2c\psi(-1)M^2(2 - i\alpha\beta M^2 c)}{(1 - i\alpha\beta M^2 c - \varkappa\alpha^2 M^2 c^2)(M^2(1 + \varkappa\alpha^2)c^2 + i\alpha\beta M^2 c - 1} - 4/c\right)/(1 + \psi(-1)),$$

$$\Delta = \frac{\alpha^2 M^2 c(2 - i\alpha\beta M^2 c)}{(1 - i\alpha\beta M^2 c - \varkappa\alpha^2 M^2 c^2)^2}.$$

The eigenvalues c corresponding to the perturbations symmetrical in v (mode I) were found from a solution of the boundary-value problem for ψ. For antisymmetrical perturbations (mode II) the equation was solved for φ. The method of obtaining the solutions was analogous to that described in [2, 3].

For $M = 0$, $\alpha \geq 0.2$ the resultant values of c coincide with the results of [5], in which c was given to three significant figures for $n = 2$. Certain differences for smaller α are evidently due to the fact that the error in the determination of c introduced by the transfer of the boundary conditions from infinity to a finite distance l increases with decreasing α. The substitution of the independent variable (2.1) (by analogy with [6]) enables this disadvantage to be avoided.

Figure 1 shows the spectra of eigenvalues for mode I with $M = 0$. As in the subsequent figures, curves 1-3 represent the dependence of c_i and curves 4-6, that of c_r, on the wave number α for $n = 2$, 4, and 6, respectively. We see that with increasing n, the value of c_r diminishes, while the flow becomes more unstable in the region of large α and more stable in the region of small α.

Figure 2 illustrates the dependences of c_i and c_r on α for mode II with M=0. Here as n increases the phase velocities c_r increase, while the behavior of c_i relative to α is qualitatively the same as in the first mode. The fall in c_i with increasing n in the region of small α agrees with the results of [7], in which it was shown that for long-wave perturbations

$$c_i \sim i\alpha^{1/2}\left\{\frac{1}{2}\int V^2 dy\right\}^{1/2}.$$

The compressibility of the medium also has a considerable effect on the stability of jet flow. In a compressible medium c_r rises for mode I and falls for mode II. In both cases c_i falls for large α and the range of instability with respect to α contracts. Figure 3 shows the behavior of the eigenvalues for M=1 and mode II. These results agree closely with the data of an earlier paper [8] which considered the profile (1.4) for n=1 and M=1. For small α (Table 1, which gives the data for n=6) c_i increases, the range of α for which the compressibility has a destabilizing influence on the flow being greater for mode I than for mode II. Calculations show that for a jet with smaller momentum (greater n) and a specified Mach number the range of destabilization increases.

Let us consider the action of dispersive and dissipative effects. Although the influence of these on the flow stability is much less than that of compressibility, it is interesting to consider in which sense the imaginary part of c varies.

Table 2 shows the eigenvalues for mode I and M=1, 2 with n=6. The upper and lower numbers for a specified α correspond to the real and imaginary parts of c. For comparison, column 1 gives the spectrum of eigenvalues for $\beta=\varkappa=0$; column 2, for $\beta=0$, $\varkappa=0.2$; and column 3, for $\beta=0.5$, $\varkappa=0$. We see that in a conservative medium dispersion leads to a fall in c_i with increasing α, while the phase velocity of the perturbations increases. It follows from the results presented in column 3 that dissipation makes mode I less stable. If the jet has a large momentum (n=2), dissipative effects have a stronger influence on the instability of the flow, while the stabilizing influence of dispersion starts operating at smaller wave numbers than in the case of n=6.

Table 3 illustrates the eigenvalues for mode II with M=1.2 and n=6. Columns 1-3 here correspond to the same values of the parameters as in Table 2. In this case the dispersive and dissipative effects promote greater flow stability; for smaller n their influence diminishes.

The authors wish to thank V. E. Nakoryakov for valuable comments and discussion of the results.

LITERATURE CITED

1. V. E. Nakoryakov, V. V. Sobolev, and I. R. Shreiber, "Long-wave perturbations in a gas–liquid medium," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1972).
2. V. V. Sobolev and O. Yu. Tsvelodub, "Rayleigh stability of hydrodynamic flows in a medium with dispersion," in: Wave Processes in Two-Phase Systems [in Russian], Inst. Teplofiz. Sibirsk. Otd. Akad. Nauk SSSR, Novosibirsk (1975).
3. V. E. Nakoryakov, V. V. Sobolev, and O. Yu. Tsvelodub, "Rayleigh stability of shear flow in relaxing media," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1976).
4. L. N. Howard, "Note on a paper of John W. Miles," J. Fluid Mech., 10, Pt. 4, 509-512 (1961).
5. P. G. Drazin and L. N. Howard, "Hydrodynamic stability of parallel flow of inviscid fluid," Adv. Appl. Mech., 9, 1-89 (1966).
6. A. Michalke, "On the inviscid instability of the hyperbolic-tangent velocity profile," J. Fluid Mech., 19, Pt. 4, 543-556 (1964).
7. P. G. Drazin and L. N. Howard, "The instability to long waves of unbounded parallel inviscid flow," J. Fluid Mech., 14, Pt. 2, 257-283 (1962).
8. W. Blumen, "Jet flow instability of an inviscid compressible fluid," J. Fluid Mech., 46, Pt. 4, 737-747 (1971).